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Mean First-Passage Time in a Bistable Kinetic Model with Stochastic Potentials

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The transient properties of a bistable system with the stochastic potentials are investigated. The explicit expressions of the mean first-passage time (MFPT) are obtained by using a steepest-descent approximation. The results show that the MFPT of the system increases with the amplitude Δ of the stochastic potential, decreases with the noise intensity D and the correlation length l. The stochastic potential makes against the particle moving towards the destination.

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1. INTRODUCTION

For some years the problem of noise-induced transport has attracted much interest in theoretical as well as experimental physics (Ai *et al.*, 2003a,b, 2004; Jia and Li, 1996; Reimann, 2000). This subject was motivated by the challenge to explain unidirectional transport in biological systems, as well as their potential technological applications ranging from classical nonequilibrium models to quantum systems (Ai *et al.*, 2005; Astumian and Derenyi, 1998; Astumian and Hanggi, 2002; Doering *et al.*, 1995; Maddox, 1994; Thomas and Thornhill, 1998). The bistable systems driven by the noise is one of the simplest problems. The steady-state statistical properties of a bistable kinetic model with all kinds of noises are mainly investigated (Hersthemke and Lefever, 1984; Wu *et al.*, 1994).

However, the MFPT is another important quantity in the noise-induced bistable systems. In recent years several papers have dealt with the derivation of exact expressions for the MFPT of the bistable system (Behn *et al.*, 1993; Jia and Li, 1996; Kus and Wodkiewicz, 1993; Laio *et al.*, 2001; Mei *et al.*, 1999; Porra and Lindenberg, 1995; Venkatesh and Patnaik, 1993). Behn and his co-workers

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(Behn *et al.*, 1993)studied the MFPT of the system driven by a superposition of suitably scaled independent dichotomous Markovian processes with natural boundary condition. Jia and his co-workers (Jia and Li, 1996) investigated the properties of bistable system with correlations between additive and multiplicative noise terms. The explicit expressions of the MFPT were also obtained and the MFPT is affected by the strength of correlation between the noises. The MFPT of the bistable system driven by cross-correlated noises were also investigated by Mei *et al.* (1999). The previous works on MFPT are limited to case of the deterministic potential. The present study extends the study of the MFPT to the case of the stochastic potentials. The MFPT increases with the noise intensity and the correlation length, while it decreases with the amplitude of the stochastic potential. Our emphasis is on finding stochastic effects of the potential on MFPT. The analytical expression of MFPT is achieved by using a steepest-descent approximation.

2. MEAN FIRST-PASSAGE TIME

Consider a Brownian particle moving in a bistable kinetic system, which contains a stochastic potential. The particle motion satisfies the dimensionless Langevin equation

$$\dot{x} = -U^{'}(x) + \xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise with zero mean (Risken, 1984), and

$$\langle \xi(t)\xi(t^{'}) \rangle = 2D\delta(t - t^{'}), \tag{2}$$

where D is the noise intensity. The potential contains two parts: the deterministic part $U_0(x)$ and the stochastic part $\eta(x)$

$$U(x) = U_0(x) + \eta(x). \tag{3}$$

The deterministic part $U_0(x)$ is a bistable potential

$$U_0(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{4}$$

The stochastic part $\eta(x)$ fluctuates with the position x (Jia *et al.*, 2001; Parris *et al.*, 1997; Zhao *et al.*, 2002). For the single dichotomous potential, we get

$$\eta(x) = \Delta(-1)^{n(x,0)}. \tag{5}$$

The random function $n(x, 0)$ counts the number of jumps in the interval of $(0, x)$, which satisfies that

$$\overline{n(x_2, x_1)} = \frac{|x_2 - x_1|}{l}, \tag{6}$$

where the correlation length l is the mean distance between jumps. It is easy to verify that $\overline{\eta(x)} = 0$, and the probability distribution function of $n(x, 0)$ satisfies the Possionian distribution:

$$p(n) = \frac{\exp(-\overline{n})\overline{n}^n}{n!}. \tag{7}$$

It is straightforward to calculate the spatial correlation function for the dichotomous potential

$$\overline{\eta(x_1)\eta(x_2)} = \Delta^2 \exp\left(\frac{-2|x_1 - x_2|}{l}\right). \tag{8}$$

It is noticed that

$$\begin{aligned} \exp\left[-\frac{\eta(x)}{D}\right] &= \sum_{n}^{\infty}(-1)^n \frac{1}{n!}\left(\frac{1}{D}\right)^n [\eta(x)]^n, \\ &= \cosh\left(\frac{\Delta}{D}\right) - \frac{\eta(x)}{\Delta}\sinh\left(\frac{\Delta}{D}\right). \end{aligned} \tag{9}$$

So

$$\left\langle \exp\left[-\frac{\eta(x)-\eta(x+y)}{D}\right]\right\rangle_\eta = \left[\cosh^2\left(\frac{\Delta}{D}\right) - \exp\left(-\frac{2y}{l}\right)\sinh^2\left(\frac{\Delta}{D}\right)\right]. \tag{10}$$

For the case of Ornstein-Uhlenbeck Potential, $\eta(x)$ is a sum of N independent dichotomous potentials (Fox, 1986; Gardiner, 1983; Jia and Li, 1996; Hanggi *et al.*, 1984)

$$\eta(x) = \sum_{i}^{N} \eta_i(x), \tag{11}$$

$$\left\langle \exp\left[-\frac{\eta(x)-\eta(x+y)}{D}\right]\right\rangle_\eta = \exp\left\{\left(\frac{\Delta}{D}\right)^2\left[1 - \exp\left(-\frac{2y}{l}\right)\right]\right\}. \tag{12}$$

Equation (1) has two stable states $x_1 = -1$, $x_2 = 1$ and an unstable state $x_0 = 0$.

The stationary probability distribution of the system can be obtained

$$P_{st}(x) = \frac{N}{\sqrt{D}}\exp\left[-\frac{U(x)}{D}\right]. \tag{13}$$

We now proceed with the MFPT. The exact expression for the MFPT for a particle to reach the final point x_0, from the initial point x_1 is given by Fox (1986), Gardiner (1983), Jia and Li (1996), Hanggi *et al.* (1984)

$$T(x_1 \to x_0) = \int_{x_1}^{x_0} \frac{dx}{DP_{st}(x)} \int_{-\infty}^{x} P_{st}(y)\,dy. \tag{14}$$

When the intensity of the noise D is small in comparison with the energy barrier

$$D < U(x_0) - U(x_1), \tag{15}$$

$T(x_1 \to x_0)$ becomes independent of the initial condition. With a steepest-descent approximation (Fox, 1986; Gardiner, 1983; Jia and Li, 1996; Hanggi *et al.*, 1984) Eq. (14) reduces to

$$\begin{aligned} T(x_1 \to x_0) &= \frac{2\pi}{\sqrt{U_0''(x_1)U_0''(x_0)}} \exp\left[\frac{U(x_0) - U(x_1)}{D}\right], \\ &= \frac{2\pi}{\sqrt{U_0''(x_1)U_0''(x_0)}} \exp\left[\frac{U_0(x_0) - U_0(x_1)}{D}\right] \\ &\quad \times \exp\left[\frac{\eta(x_0) - \eta(x_1)}{D}\right]. \end{aligned} \tag{16}$$

For the case of simple dichotomous potential (Fox, 1986; Gardiner, 1983; Jia and Li, 1996; Hanggi *et al.*, 1984), from Eqs. (10) and (16), we have

$$T(-1 \to 0) = \sqrt{2}\pi \left[\cosh^2\left(\frac{\Delta}{D}\right) - e^{-2/l}\sinh^2\left(\frac{\Delta}{D}\right)\right] \exp\left(\frac{1}{4D}\right). \tag{17}$$

For the case of Ornstein-Uhlenbeck potential (Fox, 1986; Gardiner, 1983; Jia and Li, 1996; Hanggi *et al.*, 1984), from Eqs. (12) and (16), we have

$$T(-1 \to 0) = \sqrt{2}\pi \exp\left[\left(\frac{\Delta}{D}\right)^2 \left(1 - e^{-2/l}\right)\right] \exp\left(\frac{1}{4D}\right). \tag{18}$$

Equations (17) and (18) are the main results of this paper.

3. RESULTS AND DISCUSSION

We have obtained the explicit expressions of MFPT for the case of the single dichotomous stochastic potential and the Ornstein-Uhlenbeck potential, respectively. Because the results from these two cases are very similar, we mainly discuss the MFPT in the Ornstein-Uhlenbeck potential.

Figure 1 shows the MFPT as a function of the noise intensity D for different values of Δ at $l = 1.0$. The MFPT decreases as the noise intensity D increases. When $D \to 0$, the particle will always stay at position $x = -1$, so that the MFPT $\ln T \to \infty$. When the noise intensity D increases, the particle can easily reach the position $x = 0$ from the position $x = -1$, such that the MFPT is small. When the amplitude Δ of the stochastic part of the potential increases, the MFPT increases,

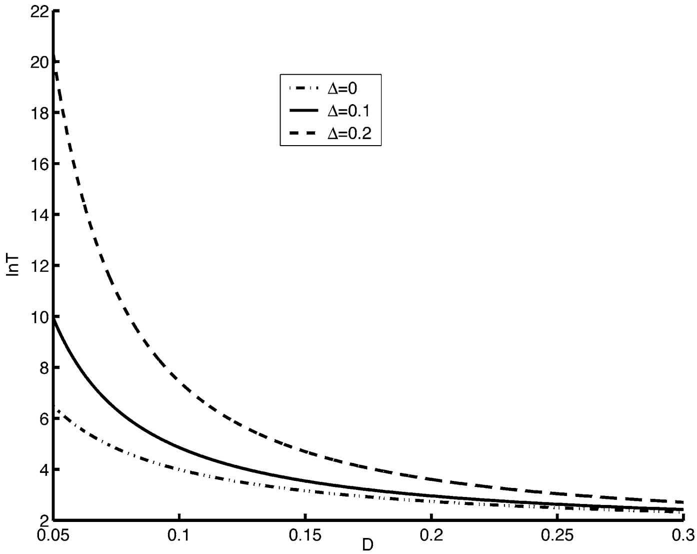

Fig. 1. MFPT $\ln T$ versus noise intensity D for different values of $\Delta = 0, 0.1, 0.2$ at $l = 1.0$.

which indicates that the stochastic potential prevent the particle moving from the position $x = -1$ to the position $x = 0$.

Figure 2 shows that the MFPT versus the amplitude Δ of the stochastic part in the potential for different values of the noise intensity D at $l = 1.0$. The MFPT increases as the amplitude Δ increases. When $\Delta = 0$, namely, there is no stochastic part in the potential, the MFPT is equal to a certain value. When the $\Delta \to \infty$, the stochastic part dominates, so that the particle can never reach the position $x = 0$ from the position $x = -1$. When the noise intensity D is small ($D = 0.1$), the change of the MFPT disappears.

Figure 3 shows the MFPT as a function of the correlation length l for different values of Δ at $D = 0.1$. The MFPT decreases as the correlation length l increases. When $l \to \infty$, namely, the mean distance between jumps, the stochastic effect in the potential disappears. When l is small, the stochastic effect increases, so it is not easy for the particle to move from the position $x = -1$ to the position $x = 0$. When $\Delta = 0$, the stochastic effect in the potential disappears, the MFPT is a constant as the correlation length l increases.

4. CONCLUDING REMARKS

We study the transient of a Brownian particle moving in a bistable system in the presence of a stochastic potential. The explicit expressions of MFPT are obtained in the two case: the simple dichotomous potential and the

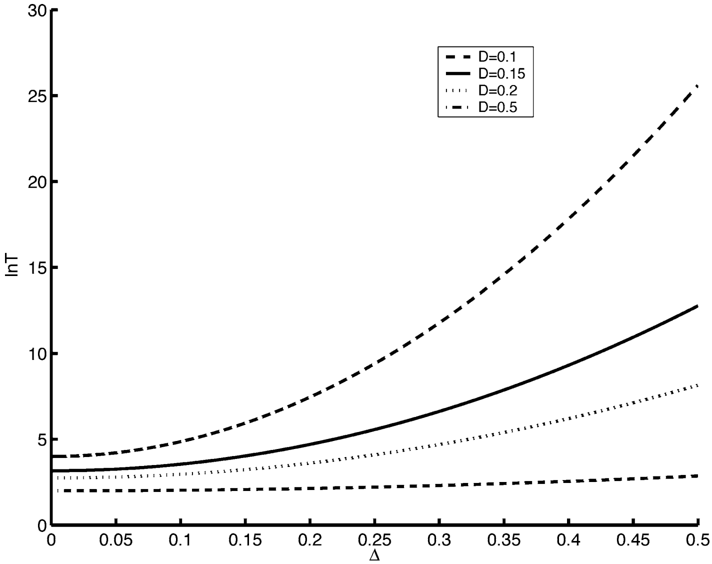

Fig. 2. MFPT $\ln T$ versus the amplitude of the stochastic potential Δ for different values of $D = 0.1, 0.15, 0.2, 0.5$ at $l = 1.0$.

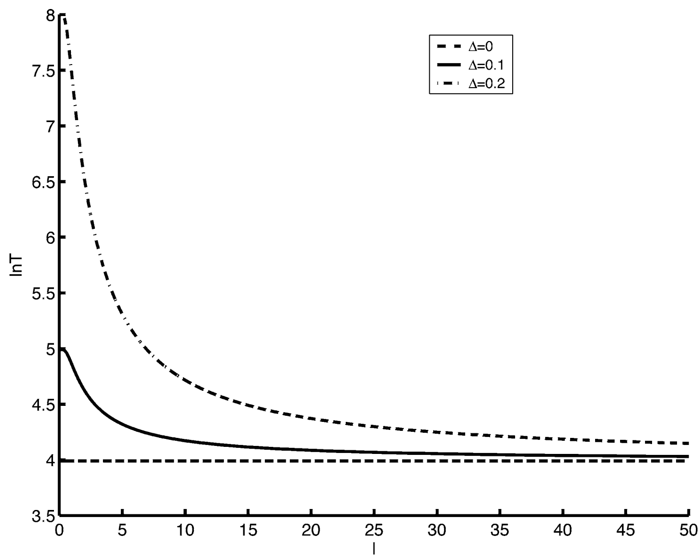

Fig. 3. MFPT $\ln T$ versus correlation length l between jumps for different values of $\Delta = 0, 0.1, 0.2$ at $D = 0.1$.

Ornstein-Uhlenbeck potential. The MFPT increases with the amplitude Δ of the stochastic part in the potential. When the stochastic part in the potential dominates, the particle will cost long time to reach one position from another potion. Oppositely, the particle will reach its destination quickly when the deterministic part dominates. The MFPT decreases as the noise intensity increases. It is obvious that the noise facilitates the diffusion of the particle. When there is no noise, the particle will always stay its initial position. When the correlation length between jumps increases, the stochastic effects of the potential decreases, the particle will reach its destination quickly and the MFPT decreases. It has been shown that the amplitude and the correlation length of the stochastic potential play opposing roles in the bistable kinetic system.

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